Rank Of Some Semigroup

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ABSTRACT

A new computational technique for rank of some semigroup is presented. The technique is based on matrix representation and simplifies the computational efforts encountered using direct definition technique of computing ranks. The important of this idea for the study of abstract groups seems to depend on the fact that group-theoretical calculations are easier to carry out in groups of matrices than in abstract groups. Its effectiveness is demonstrated in the computation of the rank of certain transformation semigroup, symmetry semigroup (S_3), Dihedral group (D_4), monogenic semigroup and the inverse semigroup. The new technique has been further employed in the computation of rank of Markov semigroup, a semigroup which admits a prefix-closed regular language of unique representatives with respect to some generating set.

Keywords: *Monoid, Transformation semigroup, Representation, Markov semigroup and Echelon matrix*

1 INTRODUCTION

The notion of 'rank', or 'dimension', belongs primarily to linear algebra. One can define the rank of a (finite-dimensional) vector space V either as the cardinality of a maximal linearly independent subset or as the cardinality of a minimal generating set of V, and it is an elementary result in linear algebra that these two cardinalities are equal.

If we try to extend the concept of rank to a more general algebraic system such as a semigroup, or indeed even a group, we find that the possible definition of rank (of which we have so far given only two) give different values.

For a semigroup S, the "classical" idea of rank is concerned with finding minimum size generating sets for S. When working with a finitely generated semigroup S determining the rank of S, denoted rank(S), is a natural consideration. The rank properties given by Howie and Ribeiro in 1999 looks very difficult in computing the rank of some semigroup, but this new method (matrix representation) is easy to carry out when one is able to list out the element of that semigroup.

1.2 Monoid: Let S be a semigroup, An element $x \in S$ is a left identity of S, if

$$\forall\, y \in S: x \cdot y = y$$

Similarly, x is a right identity of S, if

$$\forall\, y \in S: y \cdot x = y.$$

If x is both a left and a right identity of S, then x is called an identity of S. A semigroup is a monoid if it has an identity.

1.3 Homomorphisms

Let (S, ·) and (P, ⋆) be two semigroups. A mapping α: S → P is a homomorphism, if

$\forall x, y \in S : \alpha(x \cdot y) = \alpha(x) \star \alpha(y)$

Thus a homomorphism respects the product of S while 'moving' elements to P (which may have a completely different operation as its product). However, a homomorphism may also identify elements: α(x) = α(y).

1.3.1 Example (1) Let S = (N,+) and P = (N, ·), and define $\alpha(n) = 2^n$ for all n∈ N. Now,

$\alpha(n + m) = 2^{n+m} = 2^n \cdot 2^m = \alpha(n) \cdot \alpha(m)$,

and hence α: S → P is a homomorphism.

(2) Let S be the semigroup of integers S = (ℤ, ·) under multiplication, and let P be the semigroup of integers P = (ℤ,+) under addition. Define a mapping α: S → P by α(n) = n for all n ∈ℤ. Then α is *not* a homomorphism, because 6 = α(6) = α(2·3) ≠α(2) + α(3) = 5.

.1.4 The full transformation semigroup

Let X be again a set, and denote by T_X the set of all functions $\alpha: X \to X$. Then T_X is the full transformation semigroup on X with the operation of composition of functions.

1.5 Representations

A homomorphism ϕ: S → T_X is a representation of the semigroup S. We say that ϕ is a faithful representation, if it is an embedding, ϕ: S ↪T_X.

The following theorem states that semigroups can be thought of as subsemigroups of the full transformation semigroups, that is, for each semigroup S there exists a set X such that $S \cong P \le T_X$ for a subsemigroup P of transformations

1.5.1 Theorem **.** *Every semigroup* S *has a faithful representation.*

Proof**.** Let $X = S^1$, that is, add the identity 1 to S if S is not a monoid. Consider the full transformation semigroup $T = T_{S^1}$. For each $x \in S$ define a mapping $\rho_x : S^1 \to S^1, \rho_x(y) = xy\,(y \in S^1)$.

Thus $\rho_x \in T$, and for all x, y ∈ S and for all z $\in S^1$,

$\rho_{xy}\,(z) = (xy)z = \rho x(yz) = \rho_x\,(\rho_y(z)) = \rho_x\rho_y(z)$,

and hence $\rho_{xy} = \rho_x\rho_y$. Consequently, the mapping

$$\phi: S \to T, \phi(x) = \rho_x$$

is a homomorphism. For injectivity we observe that

$\phi(x) = \phi(y) \Rightarrow \rho_x = \rho_y \Rightarrow \rho_x(1) = \rho_y(1) \Rightarrow x = y$.

1.6 Word semigroups: Let A be a set of symbols, called an alphabet. Its elements are letters. Any finite sequence of letters is a word (or a string) over A. The set of all words over A, with at least one letter, is denoted by A^+. For clarity, we shall often write u ≡ v, if the words u and v are the same (letter by letter).

The set A^+ is a semigroup, the word semigroup over A, when the product is defined as the catenation of words, that is, the product of the words $w_1 \equiv a_1 a_2 \ldots a_n, w_2 \equiv b_1 b_2 \ldots b_m$ $(a_i, b_i \in A)$ is the word $w_1 \cdot w_2 = w_1 w_2 \equiv a_1 a_2 \ldots a_n b_1 b_2 \ldots b_m$.
When we join the empty word 1 (which has no letters) into A^+, we have the word monoid A^*, $A^* = A^+ \cup \{1\}$. Clearly, $1 \cdot w = w = w \cdot 1$ for all words $w \in A^*$.
1.6.1 *Example* . Let $A = \{a, b\}$ be a binary alphabet. Then $a, b, aa, ab, ba, bb, aaa, aab, \ldots$ are words in A^+. Now, $ab \cdot bab \equiv abbab$. As usual, w^k means the catenation of w with itself k times, and so, for example, $v \equiv ab^3(ba)^2 \equiv abbbbaba \equiv ab^4aba$.
1.6.2 Free semigroups

Let S be a semigroup. A subset $X \subseteq S$ generates S freely, if $S = [X]_S$ and every mapping $\alpha_0 : X \rightarrow P$ (where P is any semigroup) can be extended to a homomorphism $\alpha: S \rightarrow P$ such that $\alpha \restriction X = \alpha_0$. Here we say that α is a homomorphic extension of the mapping α_0. If S is freely generated by some subset, then S is a free semigroup.
1.6.3 *Example* . (1) $(\mathbb{N}_+,+)$ is free, for $X = \{1\}$ generates it freely: Let $\alpha_0 : X \rightarrow P$ be a homomorphism, and define $\alpha: \mathbb{N}_+ \rightarrow P$ by $\alpha(n) = \alpha_0(1)^n$. Now, $\alpha \restriction X = \alpha_0$, and α is a homomorphism: $\alpha(n + m) = \alpha_0(1)^{n+m} = \alpha_0(1)^n \cdot \alpha_0(1)^m = \alpha(n) \cdot \alpha(m)$.
(2) On the other hand, $(\mathbb{N}_+, \cdot)$ is *not* free. For, suppose $X \subseteq \mathbb{N}_+$, choose P = $(\mathbb{N}_+,+)$, and let $\alpha_0(n) = n$ for all $n \in X$. If $\alpha: (\mathbb{N}_+, \cdot) \rightarrow P$ is any homomorphism, then $\alpha(n) = \alpha(1 \cdot n) = \alpha(1) + \alpha(n)$, and thus $\alpha(1) = 0 \notin P$. So certainly no α can be an extension of α_0.
1.7 Generating Sets: Let X be a subset of a semigroup S, we say that the set X generates S as a semigroup if every element of S can be written as a product of element of X. Let X be the subset of an inverse semigroup I, we say that the set X generate I as an inverse if the set XUX^{-I} generates I as a semigroup where X^{-I} is the set of inverses (in the semigroup theoretic sense) of element of X. Let X be the subset of a monoid M, we say that X generates M as a monoid if the set $Xu\{1_M\}$ generates M as a semigroup, where 1_M is the identity of M. A set which generate a semigroup S as a semigroup is called a semigroup generating set for S.

A semigroup which can be generating as a semigroup by a finite set is called finitely generated.
1.8 Definitions: Howie and Ribeiro (1999), introduced five different type of rank for semigroups. These ranks $r_1(S)$, $r_2(S)$, $r_3(S)$, $r_4(S)$ and $r_5(S)$, are defined as follows:

- $r_1(S) = \max\{k$: every subset U of S cardinality k is independent$\}$
- $r_2(S) = \min\{k$: there exists a subset U of S cardinality k such that U generates $S\}$
- $r_3(S) = \max\{k$: there exists a subset U of S cardinality k which is independent and generates $S\}$
- $r_4(S) = \max\{k$: there exists a subset U of S cardinality k which is independent$\}$
- $r_5(S) = \min\{k$: every subset U of S cardinality k generates $S\}$

It has been proved that $r_1(S) \leq r_2(S) \leq r_3(S) \leq r_4(S) \leq r_5(S)$, and for convenience, the terminology has been used as $r_1(S)$ is small rank, $r_2(S)$ is lower rank, $r_3(S)$ is intermediate rank, $r_4(S)$ is upper rank and $r_5(S)$ is large rank. Here, the lower rank is what is normally called the *rank*, which has been extensively studied.

1.9 Echelon form: A matrix that has undergone Gaussian elimination is said to be in row echelon form or, more properly, "reduced echelon form" or "row-reduced echelon form." Such a matrix has the following characteristics:

1. All zero rows are at the bottom of the matrix

2. The leading entry of each nonzero row after the first occurs to the right of the leading entry of the previous row.

3. The leading entry in any nonzero row is 1.

4. All entries in the column above and below a leading 1 are zero.

Another common definition of echelon form only requires zeros below the leading ones, while the above definition also requires them above the leading ones.

1.10 Matrix representation

The theory of group representation is concerned with the problem of classifying homomorphisms of an abstract finite group into groups of matrices or linear transformation.

A matrix representation of a group G of degree n is homomorphism

$T: g \rightarrow T(g)$ of G into GL(n, k)

Two matrix representation T and T^I are equivalent if they have the same degree, say n, and if there exists a fixed matrix M in GL(n, k) such that

$T^I(g) = MT(g)M^{-I} \qquad (g \in G)$

If T is a representation of G with space S, then from the homomorphism proprety we have

$$T(ab) = T(a)T(b), \qquad (a, b \in G$$
$$T(a)^{-1} = T(a^{-1}),$$
$$T(1) = 1_S$$

Where 1_S denotes the identity mapping on S. The corresponding statement holds, of course, for matrix representations.

Example is the permutation representation of a group

1.11Markov Semigroup

Examine the interaction of Markov semigroups with adjoining identities and zeros, with direct products, with free products, and with finite index subsemigroups and extensions. Finally, the class of languages that are Markov languages for semigroups is considered in [2].

Since the study of Markov semigroups seems to be an entirely new area, there are many possible directions for further research. Consequently, various language theory.

The different rank properties described by Howie and Ribeiro in 1999 and the difficulties in computing each rank inspired this alternative approach in computing rank. We shall construct a cayley tabel for each of the semigroup. Transform it into matrix representation form, and finally to Echelon form and then compute the rank.

2 MAIN RESULT

(1) The rank of the upper triangular integer matrices

$$S = \{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \geq 1\}$$

is 4

(2) The rank of a semigroup $S = \{o, e, f, a, b\}$ where

$$O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\ a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is 2

(3) Let X ={1, 2, 3}. A mapping $\alpha : X \rightarrow X$ which is defined by

$\alpha(1) = 2,\ \alpha(2) = 3, and\ \alpha(3) = 3$

that is for

$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$ where S=$\langle \alpha, \beta \rangle_{T_X}$ is the subsemigroup of T_X $\alpha\ and\ \beta$ generated by $\alpha\ and\ \beta$. The rank is 4

(4) The dihedral group has rank equal 6

2.1 Considering mealy automaton construction the rank associated with the markov semigroup of that state function is 1 when the input and output are different and 2 when they are the same.

3 DETAILS

The word "rank" refers to several related concepts in mathematics involving graphs, groups, matrices, quadratic forms, sequences, set theory, statistics, and tensors.

The rank of a mathematical object is defined whenever that object is free. In general, the rank of a free object is the cardinality of the free generating subset G.

Cayley table of some semigroups and their matrix representations

The following examples of semigroups and their matrix representation are used for computation:

1. A set S= $Z_6 = (0, 1, 2, 3, 4, 5)$ is a semigroup.
   The cayley table for (S, +) and (S, **.**) is as follows

+	0 1 2 3 4 5
0	0 1 2 3 4 5
1	1 2 3 4 5 0
2	2 3 4 5 0 1
3	3 4 5 0 1 2
4	4 5 0 1 2 3
5	5 0 1 2 3 4

•	0 1 2 3 4 5
0	0 0 0 0 0 0
1	0 1 2 3 4 5
2	0 2 4 0 2 4
3	0 3 0 3 0 3
4	0 4 2 0 4 2
5	0 5 4 3 2 1

Notice that for (S, +) the generator is 1, so that matrix representation for 1 is

$$R(1) = \begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \end{pmatrix}$$

For (S, **.**), the matrix representation is

$$R(1) = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \end{pmatrix}, \quad R(2) = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&1&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \end{pmatrix}$$

$$R(3) = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \end{pmatrix}, \quad R(5) = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix}$$

2. Let S = $\{o, e, f, a, b\}$ be the semigroup where

$O = \begin{pmatrix} 0&0 \\ 0&0 \end{pmatrix}$, $a = \begin{pmatrix} 0&1 \\ 0&0 \end{pmatrix}$, $b = \begin{pmatrix} 0&0 \\ 1&0 \end{pmatrix}$, $e = \begin{pmatrix} 1&0 \\ 0&0 \end{pmatrix}$, $f = \begin{pmatrix} 0&0 \\ 0&1 \end{pmatrix}$

The cayley table is as follows:

•	o e f a b
0	0 0 0 0 0
e	o e o a o
f	o o f o b
a	o o a o e
b	o b o f o

Notice that e = ab, b^2 = 0 = a^2

= fa = ef = bf = fe and f = ba

a and b generate S.

The matrix representation R of the above table is

$$R(a) = \begin{pmatrix} 0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0 \end{pmatrix}, \quad R(b) = \begin{pmatrix} 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0 \end{pmatrix}$$

3. The group D_4 consists of elements

$\{(1), (1234), (13)(24), (14)(23), (1432), (12)(34), (24), (13)\}$

To construct a cayley table I will like to rearranged my group D_4 as follows

D_4 = X = $\{e, a, b, c, h, k, r, s\}$ such that

E = (1), a = (1432), b = (13) (24), c = (1234), h = (12) (34), k = (14) (23), r = (24), S = (13)

The table is as follows;

O	e a b c h k r s
e	e a b c h k r s
a	a b c e s r h k
b	b c e a k h s r
c	c e a b r s k h
h	h r k s e b a c
k	k s h r b e c a
r	r k s h c a e b
s	s h r k a c b e

(X, o) is a semigroup

Since composition of permutation is

Associative.

The matrix representation R of D_4 is

$$R(c) = \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{pmatrix}, \quad R(k) = \begin{pmatrix} 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \end{pmatrix}$$

4. The set H = $\{(1), (1234), (13)(24), (1432)\}$ is a subgroup of D_4 and the cayley table is shown below:

*	(1)	(1234)	(13) (24)	(1432)
(1)	(1)	(1234)	(13) (24)	(1432)
(1234)	(1234)	(13)(24)	(1432)	(1)
(13) (24)	(13) (24)	(1432)	(1)	(1234)
(1432)	(1432)	(1)	(1234)	(13) (24)

(H, *) is a semigroup

The matrix representation R is

$$R(1234) = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}$$

4 Rank of the matrix representation

We shall use the following notations; R for matrix representation, r for rank and ER for echelon form of the matrix representation.

1.

$$R(1) = \begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \end{pmatrix} \text{ for } (S, +), \quad ER(1) = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}$$

r(1) = 6
then for the semigroup (S,**.**)

$$R(1) = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \end{pmatrix}, \qquad R(2) \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \end{pmatrix}$$

$$R(3) = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&1&0&0 \end{pmatrix}, \qquad R(5) = \begin{pmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix}$$

$$ER(1) = \begin{pmatrix} 0&1&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \qquad ER(2) = \begin{pmatrix} 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$$

$$ER(3) = \begin{pmatrix} 0&1&0&1&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}, \qquad ER(5) = \begin{pmatrix} 0&1&0&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}$$

R(1) = 2, r(2) = 3, r(3) = 1, r(5) = 2

2. $$ER(1234) = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$$

r(1234) = 4

3. Consider the group consisting of the elements where
G = $\{e, x, x^2, y, xy, x^2y, x^3 = e = y^2,\ xy = yx^2\}$

The cayley table is as shown below

	e	x	x^2	y	xy	x^2y
e	e	x	x^2	y	xy	x^2y
x	x	x^2	e	xy	x^2y	y
x^2	x^2	e	x	x^2y	y	xy
y	y	x^2y	xy	e	x^2	x
xy	xy	y	x^2y	x	e	x^2
x^2y	x^2y	xy	y	x^2	x	e

The matrix representations:

$$R(e) = \begin{pmatrix} 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{pmatrix}, \quad R(x) = \begin{pmatrix} 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \end{pmatrix}, \quad R(y) = \begin{pmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \end{pmatrix}$$

x and y generates the semigroup and so the representation of x and y is enough. The echelon form and the rank are equal.

That is

$r(e) = r(x) = r(y) = 6$

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